

A MATHEMATICAL MODEL FOR THE LINEAR DYNAMIC BEHAVIOR OF TWO PHASE PERIODIC MATERIALS

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Abstract—In this study, a mathematical model is developed for two phase materials with the object of using it for predicting the response of masonry walls to dynamic inputs. The method employed here uses the theory of mixtures applied to a two phase material in which the phases reflect a periodic structure and in which each phase is linearly elastic. Employing the fundamental equations of the theory of mixtures, the governing equations of a linear approximate theory are established. The theory, valid for an arbitrary direction of motion, replaces the composite by a homogeneous, two phase, anisotropic, elastic solid. It accommodates the dispersive nature of the composite by means of an elasto-dynamic operator, which is introduced into the constitutive relations of the linear momentum interactions.

1. INTRODUCTION

With the motivation of studying the response of masonry walls to dynamic inputs, in this work we develop a mathematical model for two phase materials. We assume that both of the phases are linearly elastic and perfectly bonded at their interfaces and that the phases have a periodic structure.

In establishing the mathematical model, several approaches can be adopted. The first is the exact treatment which includes the field equations of elasticity for each phase and the equations of continuity at the interfaces. As this approach makes the analysis very complicated, it is not of practical interest, and thus the development of an approximate theory becomes a necessity.

During the last few years, a number of approximate theories have been proposed. In the first of these, the two phase composite is replaced by a homogeneous, anisotropic, elastic medium. As this theory, called effective modulus theory, does not accommodate any dispersion, it is valid only when the wave length is very large. As an example of this type of theory, we refer to a study by Rytov [1], where an effective modulus theory is developed for a layered composite. To compensate for the shortcomings of the effective modulus theory, another approximate theory, called effective stiffness theory, is proposed for layered and fiber reinforced composites in Refs. [2-4]. In this theory the approximate governing equations are obtained by expanding the displacements for each constituent in power series and introducing the series into a variational functional. However, the possibility of extending this theory to composites containing vertical layering does not appear to be fruitful, as the theory has rather complicated equations even for simple composites like layered and fiber reinforced materials. Hegemier *et al.* [5,6] proposed another approach, which they called theory of interacting continua. The theory contains a micro-structure and is based on asymptotic expansions of the field variables with respect to the space variables. However, they developed the theory again for layered and fiber reinforced composites only. For waves propagating perpendicular to layering in a layered material, the exact spectrum has a banded structure with passing and stopping bands, which contain points governing harmonic waves which are propagated and attenuated respectively. Herrman, Kaul and Delph [7, 8] have developed a one-dimensional approximate theory which they call effective dispersion theory, for waves propagating perpendicular to layering only. Their theory accommodates the first stopping band and approximates quite well the two lowest spectral lines over the first two Brillouin zones.

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We adopt a different approach for establishing the mathematical model for two phase materials. In this study the material is considered as a mixture consisting of two phases and the theory of mixtures is used to obtain approximate equations governing its dynamic behavior. The resulting theory is a general one which would include the geometry of a masonry wall as a particular case. The only restrictions imposed by the theory are that the two phases exhibit a periodic form and that the material of each phase be linearly elastic.

The idea of treating composites as a mixture is not new. In fact, it has previously been used in Refs. [9, 10] for developing approximate theories predicting the dynamic response of layer and fiber reinforced composites when the motion is in the direction of the layers of fibers. Later, Bedford [11] attempted to extend this theory to a general case where the motion could have an arbitrary direction. However, his method has some shortcomings as he did not take into account coupling in the stress-strain relations which would imply that the state of stress of one phase is affected by the deformation of the other.

The approximate theory we propose in this study for an arbitrary direction of motion replaces the heterogeneous two phase composite by an homogeneous, anisotropic, elastic solid. In developing the theory, we have chosen the mixture approach for several reasons. First, this method leads to equations simple enough to be used in the dynamic analysis of complicated composite materials such as masonry walls. Secondly, since we account for linear momentum interactions between the phases, the resulting approximate equations not only exhibit anisotropy, but also accommodate dispersion. The dispersion is accounted for by a time dependent operator, which we call an elastodynamic operator. It relates the interaction forces to the difference of the average phase displacements. The introduction of this operator, which we believe is new, not only improves the matching of approximate fundamental spectral lines with exact ones, but also makes it possible to match the second approximate and exact cut-off frequencies. Third, the mixture approach allows us to write the approximate equations of all two phase composites in a common form. The difference comes when we adapt this form to a particular geometry. The microstructure derived from the geometry governs specific forms of the interaction and stress constitutive equations of a given composite.

The total study of the theory is, of necessity, very lengthy so that we have had to divide the study into three parts in reporting on it. In this first paper the general theory for two phase materials is developed. To make the paper more comprehensive we discuss in a general way the further development, which follows in a second paper, when a particular periodic structure is specified.

The second paper is divided into two parts. In the first, the general theory is adapted to a material that is made up of periodic layers. This choice is made as the geometry approximates that of masonry walls, and because there are available for layered materials an exact theory and a large amount of experimental data describing the propagation of waves in specified directions. The development from the general to the particular is significant and ends with a particular model containing an array of constants. The second part of the paper is devoted to establishing these constants. The way in which they are found is by no means unique, and only the method chosen is described in detail. The final part of the second paper compares the behavior of infinite trains of waves in the layered media with predictions from the exact theory and certain experimental results.

In the third paper the completed model is assessed. Transient wave behavior in a number of directions predicted by the theory is compared with experimental data.

2. FUNDAMENTAL EQUATIONS

As we adopt the theory of mixtures for developing the theory, we first review the fundamental equations of a two phase mixture, which have been studied extensively by many researchers (see, e.g. [12-17]). All the variables appearing in these equations are related to the average values of the quantities they represent.

The local forms of the fundamental equations for a two phase composite, which is referred to an (x_1, x_2, x_3) Cartesian coordinate system, are:

Conservation of mass

$$\frac{\partial \rho_\alpha}{\partial t} + \frac{\partial}{\partial x_i} (\rho_\alpha v_i^\alpha) = 0. \quad (2.1)$$

Equations of linear momentum

$$\rho_\alpha \frac{D^\alpha v_i^\alpha}{Dt} = \frac{\partial \sigma_{ij}^\alpha}{\partial x_j} + \rho_\alpha F_i^\alpha + M_i^\alpha. \quad (2.2)$$

Balance of energy

$$\rho_\alpha r^\alpha - \frac{\partial q_i^\alpha}{\partial x_i} - \rho_\alpha \frac{D^\alpha \epsilon^\alpha}{Dt} + \sigma_{ki}^\alpha \frac{\partial v_i^\alpha}{\partial x_k} + \psi^\alpha = 0. \quad (2.3)$$

The Clausius–Duhem inequality

$$\rho_\alpha \frac{D^\alpha S^\alpha}{Dt} - \rho_\alpha \frac{r^\alpha}{T^\alpha} + \frac{\partial}{\partial x_k} \left(\frac{q_k^\alpha}{T^\alpha} \right) + \beta^\alpha \geq 0. \quad (2.4)$$

We note that in writing the equation of conservation of mass (eqn 2.1), we neglect the mass transfer between the phases. We assume that there is no angular momentum interaction between the phases. This implies that the partial stress components, σ_{ij}^α , are symmetric. This assumption is consistent with the classical mixture theory which is used in this study. To take into account the angular momentum interaction between phases, one needs to use the micro-morphic mixture theory by which antisymmetric distributions of the field variables can be accommodated.

In accordance with the equations of the theory of mixtures [12–17], the interactions M_i^α , ψ^α and β^α satisfy the relations

$$\sum_{\alpha=1}^2 M_i^\alpha = 0; \quad \sum_{\alpha=1}^2 (M_i^\alpha v_i^\alpha + \psi^\alpha) = 0; \quad \sum_{\alpha=1}^2 \beta^\alpha = 0. \quad (2.6)$$

If the mass densities of the phases are denoted by ρ_α^R , the partial masses ρ_α are related to ρ_α^R by

$$\rho_\alpha = n_\alpha \rho_\alpha^R, \quad (2.7)$$

where n_α is the volume fraction of the α -phase with the property

$$\sum_{\alpha=1}^2 n_\alpha = 1. \quad (2.8)$$

Finally, an important comment regarding the Clausius–Duhem inequality is in order. The form of the Clausius–Duhem inequality for a mixture is still the subject of controversy. Even though the form (eqn 2.4), used in this study is physically acceptable for two phase composites where the constituents are separate in microscale, it is by no means universally accepted.

3. THERMODYNAMIC ANALYSIS FOR INFINITESIMAL DEFORMATIONS

In the literature, thermodynamic analysis is presented in general terms for various mixtures composed of nonlinear phases (see, e.g. [15–17]). In these works a single common temperature is assumed for all phases. Here we present thermodynamic analysis for our specific mixture, i.e. for linear two phase composites, by using the Coleman–Noll procedure. Through this analysis, we establish the specific form of the stress constitutive relations and heat conduction equations for our particular material. Our analysis is based on the fundamental equations presented in the previous section and on the findings established in the area of the theory of mixtures. We assume that deformations are infinitesimal and both phases are elastic.

We begin the analysis by approximating the operator (D^α/Dt) by

$$\frac{D^\alpha}{Dt} \cong \frac{\partial}{\partial t}$$

for infinitesimal theory. If internal energy and entropy densities, and heat rates due to heat sources are redefined by

$$\dot{E}^\alpha = \rho_\alpha \dot{\epsilon}^\alpha; \quad R^\alpha = \rho_\alpha r^\alpha; \quad \dot{\eta}^\alpha = \rho_\alpha \dot{S}^\alpha$$

per unit volume of the composite, the energy equation (2.3), and the Clausius–Duhem inequality, (2.4), become

$$R^\alpha - \frac{\partial q_i^\alpha}{\partial x_i} - \dot{E}^\alpha + \sigma_{ki}^\alpha \dot{e}_{ki}^\alpha + \psi^\alpha = 0 \quad (3.1)$$

$$T^\alpha \dot{\eta}^\alpha - R^\alpha + \frac{\partial q_k^\alpha}{\partial x_k} - \frac{q_k^\alpha}{T^\alpha} \frac{\partial T^\alpha}{\partial x_k} + T^\alpha \beta^\alpha \geq 0, \quad (3.2)$$

respectively, where the dot indicates partial differentiation with respect to time, and the e_{ij}^α , defined by

$$e_{ij}^\alpha = \frac{1}{2} \left(\frac{\partial u_i^\alpha}{\partial x_j} + \frac{\partial u_j^\alpha}{\partial x_i} \right), \quad (3.3)$$

represent the infinitesimal strain components for the α -phase. In eqn (3.3) the u_i^α are the average phase displacement components.

With the aid of the energy equation (3.1), the Clausius–Duhem inequality (3.2), takes the form

$$T^\alpha \dot{\eta}^\alpha - \dot{E}^\alpha + \sigma_{ki}^\alpha \dot{e}_{ki}^\alpha + \psi^\alpha - \frac{q_k^\alpha}{T^\alpha} \frac{\partial T^\alpha}{\partial x_k} + T^\alpha \beta^\alpha \geq 0. \quad (3.4)$$

When we write eqn (3.4) for $\alpha = 1$ and $\alpha = 2$, and add (using eqns 2.6) we get

$$\begin{aligned} \sum_{\alpha=1}^2 T^\alpha \dot{\eta}^\alpha - \dot{E} + \sum_{\alpha=1}^2 \sigma_{ki}^\alpha \dot{e}_{ki}^\alpha - \sum_{\alpha=1}^2 \frac{q_k^\alpha}{T^\alpha} \frac{\partial T^\alpha}{\partial x_k} \\ - M_i^1 (v_i^1 - v_i^2) + \beta^1 (T^1 - T^2) \geq 0, \end{aligned} \quad (3.5)$$

which is the total entropy inequality written per unit volume of the composite. In eqn (3.5)

$E = \sum_{\alpha=1}^2 E^\alpha$ describes the total internal energy density.

In accordance with the findings established in the theory of mixtures [10] we assume

$$E^\alpha = E^\alpha (e_{ij}^1, e_{ij}^2, \eta^\alpha). \quad (3.6)$$

Equation (3.6) shows that the phase internal energy density E^α depends on the deformations of both phases. This, as it will be seen later, leads to a coupling in the stress–strain relations implying that the state of stress of one phase is influenced by the deformations of the other. We believe that this coupling, which is disregarded in Ref. [11], is crucial for an adequate description of a composite as a mixture. Using eqn (3.6) and the definition of the total internal energy density, the total entropy inequality (3.5), can be written in the form

$$\begin{aligned} \sum_{\alpha=1}^2 \left(T^\alpha - \frac{\partial E}{\partial \eta^\alpha} \right) \dot{\eta}^\alpha + \sum_{\alpha=1}^2 \left(\sigma_{ij}^\alpha - \frac{\partial E}{\partial e_{ij}^\alpha} \right) \dot{e}_{ij}^\alpha \\ - \sum_{\alpha=1}^2 \frac{q_k^\alpha}{T^\alpha} \frac{\partial T^\alpha}{\partial x_k} - M_i^1 (v_i^1 - v_i^2) + \beta^1 (T^1 - T^2) \geq 0. \end{aligned} \quad (3.7)$$

Since η^α and e_{ij}^α are independent state variables, in order to satisfy this inequality we should have

$$T^\alpha = \frac{\partial E}{\partial \eta^\alpha} = \frac{\partial E^\alpha}{\partial \eta^\alpha}$$

$$\sigma_{ij}^\alpha = \frac{\partial E}{\partial e_{ij}^\alpha} = \sum_{\beta=1}^2 \frac{\partial E^\beta}{\partial e_{ij}^\beta} \quad (3.8)$$

When eqns (3.8) are taken into account the total entropy inequality reduces to

$$-\sum_{\alpha=1}^2 \frac{q_k^\alpha}{T^\alpha} \frac{\partial T^\alpha}{\partial x_k} - M_i^1 (v_i^1 - v_i^2) + \beta^1 (T^1 - T^2) \geq 0. \quad (3.9)$$

Introducing the Helmholtz free energy density ϕ^α

$$\phi^\alpha = E^\alpha - T^\alpha \eta^\alpha \quad (3.10)$$

and defining the total Helmholtz free energy density ϕ

$$\phi = \sum_{\alpha=1}^2 \phi^\alpha = E - \sum_{\alpha=1}^2 T^\alpha \eta^\alpha, \quad (3.11)$$

and using eqns (3.6) and (3.8), we obtain the relations

$$\phi^\alpha = \phi^\alpha (e_{ij}^1, e_{ij}^2, T^\alpha) \quad (3.12)$$

and

$$\eta^\alpha = -\frac{\partial \phi}{\partial T^\alpha} = -\frac{\partial \phi^\alpha}{\partial T^\alpha}$$

$$\sigma_{ij}^\alpha = \frac{\partial \phi}{\partial e_{ij}^\alpha} = \sum_{\beta=1}^2 \frac{\partial \phi^\beta}{\partial e_{ij}^\beta}, \quad (3.13)$$

which imply

$$\eta^\alpha = \eta^\alpha (e_{ij}^1, e_{ij}^2, T^\alpha)$$

$$\sigma_{ij}^\alpha = \sigma_{ij}^\alpha (e_{ij}^1, e_{ij}^2, T^1, T^2). \quad (3.14)$$

We let the temperatures of both phases be T_0 in the reference configuration of the two phase composite, which is assumed to be free of stresses. As we are dealing with an infinitesimal theory and are assuming small deviations from the reference temperature T_0 , we expand the stresses and entropies about zero deformation and the reference temperature T_0 using Taylor's formula, and retain only linear terms. When we use the symmetry conditions

$$\frac{\partial \eta^\alpha}{\partial e_{ij}^\beta} = -\frac{\partial \sigma_{ij}^\beta}{\partial T^\alpha}, \quad \frac{\partial \sigma_{ij}^\alpha}{\partial e_{mn}^\beta} = \frac{\partial \sigma_{mn}^\beta}{\partial e_{ij}^\alpha} \quad (\alpha, \beta = 1, 2), \quad (3.15)$$

which are the implications of eqns (3.13), we obtain

$$\eta^\alpha = \eta_0^\alpha + \sum_{\beta=1}^2 p_{ij}^{\alpha\beta} e_{ij}^\beta + \frac{c^\alpha}{T_0} \theta^\alpha$$

$$\sigma_{ij}^\alpha = \sum_{\beta=1}^2 C_{ijmn}^{\alpha\beta} e_{mn}^\beta - \sum_{\beta=1}^2 p_{ij}^{\beta\alpha} \theta^\beta, \quad (3.16)$$

where the coefficients $C_{ijmn}^{\alpha\beta}$ have the property

$$C_{ijmn}^{\alpha\beta} = C_{mni j}^{\beta\alpha}. \quad (3.17)$$

In eqns (3.16) θ^α , defined by

$$\theta^\alpha = T^\alpha - T_0,$$

represents phase temperature deviations from the reference temperature T_0 ; η_0^α denotes the value of the phase entropy density η^α in the reference configuration; c^α is the specific heat at constant deformation for the α -phase, measured per unit volume of the composite, and is defined by

$$c^\alpha = T_0 \left. \frac{\partial \eta^\alpha}{\partial T^\alpha} \right|_{\epsilon_{ij}^1 = \epsilon_{ij}^2 = 0, T^\alpha = T_0}.$$

The coefficients $C_{ijmn}^{\alpha\beta}$ and $p_{ij}^{\alpha\beta}$ appearing in the second of eqns (3.16) denote the material constants of a linear two phase composite. The symmetry of strain and stress components further imposes the conditions

$$\begin{aligned} p_{ij}^{\alpha\beta} &= p_{ji}^{\alpha\beta} \\ C_{ijmn}^{\alpha\beta} &= C_{jimn}^{\alpha\beta} \\ C_{ijmn}^{\alpha\beta} &= C_{ijnm}^{\alpha\beta} \end{aligned} \quad (3.18)$$

on the material constants. The conditions, eqns (3.17) and (3.18), indicate that the number of independent material constants in the stress-strain relations, the second of eqns (3.16), is at most 102.

The coefficients (C_{ijmn}^{12} , C_{ijmn}^{21}) and (p_{ij}^{12} , p_{ij}^{21}) describe the coupling in the stress constitutive equations, the second of eqns (3.16). These terms respectively permit the state of stress of one phase to be affected by the deformation and temperature deviation of the other.

We now turn our attention to deriving the heat conduction equations for a two phase composite. We first notice that the energy equation (3.1), when eqns (3.6) and (3.8) are used, reduces to

$$R^\alpha - \frac{\partial q_i^\alpha}{\partial x_i} - \sum_{\beta=1}^2 \frac{\partial E^\alpha}{\partial e_{ij}^\beta} \dot{e}_{ij}^\beta - T^\alpha \dot{\eta}^\alpha + \sigma_{ij}^\alpha \dot{e}_{ij}^\alpha + \psi^\alpha = 0. \quad (3.19)$$

We assume that partial heat flux vectors are related to the gradients of the average phase temperatures by the Fourier equation

$$q_i^\alpha = - \sum_{\beta=1}^2 k_{ij}^{\alpha\beta} \frac{\partial \theta^\beta}{\partial x_j}, \quad (3.20)$$

where $k_{ij}^{\alpha\beta}$ are the coefficients of heat conduction defined per unit area of the composite. The coefficients k_{ij}^{12} and k_{ij}^{21} in eqn (3.20) describe the thermal coupling between phases.

When the linear entropy relation, the first of eqns (3.16) and the Fourier equation (3.20), are used and when the temperature deviation from the equilibrium state is assumed to be small, the energy equation (3.19) becomes

$$R^\alpha + \sum_{\beta=1}^2 \frac{\partial}{\partial x_i} \left(k_{ij}^{\alpha\beta} \frac{\partial \theta^\beta}{\partial x_j} \right) - c^\alpha \dot{\theta}^\alpha - T_0 \sum_{\beta=1}^2 p_{ij}^{\alpha\beta} \dot{e}_{ij}^\beta + \dot{\psi}^\alpha + \psi^\alpha = 0, \quad (3.21)$$

where

$$\dot{\psi}^\alpha = \sigma_{ij}^\alpha \dot{e}_{ij}^\alpha - \sum_{\beta=1}^2 \frac{\partial E^\alpha}{\partial e_{ij}^\beta} \dot{e}_{ij}^\beta. \quad (3.22)$$

From the second of eqns (3.8) and eqn (3.22) it follows that $\dot{\psi}^\alpha$ satisfies $\sum_{\alpha=1}^2 \dot{\psi}^\alpha = 0$. Equation

(3.22) shows that, in the absence of the coupling introduced by eqn (3.6), $\dot{\psi}^\alpha$ separately vanishes for both phases. This means that the term $\dot{\psi}^\alpha$ in eqn (3.21) represents the energy interaction between the phases due to the coupling in stress constitutive equations.

We now separate the interaction due to heat exchange between the phases from the total interaction ($\psi^\alpha + \dot{\psi}^\alpha$). To do this we let

$$\psi^\alpha + \dot{\psi}^\alpha = \bar{\psi}^\alpha + Q^\alpha, \tag{3.23}$$

where Q^α represents the heat exchange between the phases (defined per unit volume of the composite) and satisfies $\sum_{\alpha=1}^2 Q^\alpha = 0$. When we substitute eqn (3.23) into eqn (3.21) we obtain the final form of the heat conduction equation

$$\sum_{\beta=1}^2 \frac{\partial}{\partial x_i} \left(k_{ij}^{\alpha\beta} \frac{\partial \theta^\beta}{\partial x_j} \right) - c^\alpha \dot{\theta}^\alpha + Q^\alpha + R^\alpha = T_0 \sum_{\beta=1}^2 p_{ij}^{\alpha\beta} \dot{e}_{ij}^{\alpha\beta} - \bar{\psi}^\alpha, \tag{3.24}$$

where the interaction term $\bar{\psi}^\alpha$ satisfies the equation

$$\sum_{\alpha=1}^2 (\bar{\psi}^\alpha + M_i^\alpha v_i^\alpha) = 0 \tag{3.25}$$

in view of the second of eqns (2.6), eqn (3.23) and the relations

$$\sum_{\alpha=1}^2 \dot{\psi}^\alpha = 0 \text{ and } \sum_{\alpha=1}^2 Q^\alpha = 0.$$

4. RESTRICTIONS ON MODEL CONSTANTS

For studying the constraints imposed on model constants we use an hypothesis which states that the strain energy function must be positive definite. For a two phase composite the strain energy function W per unit volume is defined by

$$dW = \sum_{\alpha=1}^2 \sigma_{ij}^\alpha de_{ij}^\alpha. \tag{4.1}$$

By integrating eqn (4.1) and using the second of eqns (3.16), disregarding thermal effects we get

$$W = \frac{1}{2} \sum_{\alpha=1}^2 \sum_{\beta=1}^2 C_{ijmn}^{\alpha\beta} e_{ij}^\alpha e_{mn}^\beta. \tag{4.2}$$

In the derivation of eqn (4.2) we assume that the strain energy is zero at reference configuration. Equation (4.2) indicates that the strain energy function will be positive definite only if

$$\sum_{\alpha=1}^2 \sum_{\beta=1}^2 C_{ijmn}^{\alpha\beta} f_{ij}^\alpha f_{mn}^\beta \geq 0 \tag{4.3}$$

is satisfied for all symmetric f_{ij}^α , where the equality holds only when $f_{ij}^\alpha = 0$. Equation (4.3) governs all of the constraints imposed on the model constants $C_{ijmn}^{\alpha\beta}$.

The stress constitutive equations and the positive definiteness condition can be written alternatively in matrix form by using vector representations of the stress and strain tensors. These forms are presented in Appendix A.

5. SOME REMARKS ON INTERACTION CONSTITUTIVE EQUATIONS

For completing the theory we have to supply it with additional constitutive relations for the interaction terms appearing in the linear momentum and energy equations. To this end we

postulate that the force and heat interactions between the phases are, through time dependent linear operators, related to the differences of phase displacements and temperatures respectively, i.e.

$$\begin{aligned} M_i^1 &= -M_i^2 = \Gamma_{ij}^1(u_j^2 - u_j^1) \\ Q^1 &= -Q^2 = H^1(\theta^2 - \theta^1), \end{aligned} \quad (5.1)$$

where Γ_{ij}^1 denotes the operator for linear momentum interaction and H^1 for the heat exchange. In view of Ref. [16] another term involving the gradient of phase temperatures can be added to the r.h.s. of the first of eqn (5.1). However, in order to keep the theory as simple as possible, this term representing the coupling between the linear momentum interaction and the temperature gradient is neglected. For a given composite, the forms of the operators in eqns (5.1) can be determined either experimentally or by using micro-model analysis. For example, in a subsequent study, where we will develop the theory for a two phase layered composite, approximate forms of these operators, will be established by using a procedure based on micro-model analysis, which is very similar to the one used by Biot[18,19]. He introduces a viscodynamic operator describing the friction between the fluid and solid phases of an isotropic porous material.

We believe that the inclusion of the operator Γ_{ij}^1 , which we will call the elastodynamic operator in the theory, is important. In fact, the presence of this operator in the theory, which accounts for the dispersion of waves in composites, not only brings some improvement to the matching of the approximate fundamental spectral lines with the exact, but also makes it possible to match the second approximate and exact cut-off frequencies.

We have now completed the general formulation of the linear approximate theory developed for a two phase composite. Provided that the constitutive equation for $\bar{\psi}^\alpha$ is known, the field variables u_i^α , e_{ij}^α , σ_{ij}^α and θ^α can be found by solving the linear momentum equations (3.1), strain-displacement relations (3.3), stress constitutive relations, the second of eqns (3.16), heat conduction equations (3.24), and interaction constitutive relations (5.1), subject to appropriate initial and boundary conditions.

6. COMPLETION OF THE MODEL FOR A PARTICULAR TWO PHASE MATERIAL

The approximate theory, just established, governs the dynamic response of a two phase composite. The theory not only accommodates the anisotropy of the composite, but also its dispersive characteristics caused by the interfaces separating the two phases.

When we say that the theory is complete, it must be understood that it is complete only in the general sense. The theory applies to all two phase materials with periodic structure and for materials in which each of the phases is linearly elastic. The model cannot be completed until a specific two phase material is defined, which entails specifying the geometry and the properties of each of the two phases.

The steps required in progressing from the general theory to a complete model replacing a specific two phase material are significant. They are described in detail in a paper following this, but it seems appropriate here to describe in a qualitative way what those steps entail.

When the geometry of the two phase material is known, its influence on the model constants and the elastodynamic operator must first be explored. The geometry usually imposes symmetry restrictions which significantly reduce the number of these unknowns. It remains to establish the surviving unknowns.

The best set of values for the unknowns will be achieved by using a method known as system identification. It is used in conjunction with experimental data. System identification is a systematic method for estimating the set of unknowns that will minimize the differences between responses predicted by the model and those from experiments over some specified extent of the responses.

Experimental data may not be available, and even if they are, the method of system identification could be complex and require a great deal of computer time.

Other methods are available which may be used separately or together. The first employs micro-model analysis which is based on the deformation modes assumed for the phases. When

the geometry of the two phase material is known, the micro-model can be constructed. The second method depends on having an exact theory for the two phase material. The product of the exact theory would be frequency spectra relating frequencies and wave lengths for infinite trains of waves. The unknowns could be established by matching the two sets of spectral lines for a specific set of properties of these lines. In this procedure special attention should be given to avoiding the violation of the model constraints imposed because the strain energy function is positive definite.

The approximate theory developed here is based on the theory of mixtures. With the object of keeping the theory as simple as possible, we have used the average values of the field variables. It may be that because of this simplification, an acceptable match between the responses predicted by this theory and experimental responses, or the responses from the exact theory, will not be possible. If such is the case, the theory can be improved, at the expense of complication, by employing micromorphic mixture theory. In that theory, antisymmetric distributions of the field variables are accommodated.

A last comment regarding the application of the mathematical model developed in this study to masonry walls is in order. We know that taking into account nonlinear and debonding effects in the response of masonry walls to dynamic inputs is important. The linear theory proposed in this study will form a basis to include these effects in the model. To explain this more explicitly, we refer to the second paper where a constitutive relation for the linear momentum interaction for layered composites is established through the use of micromodel analysis. This analysis is based on displacement and stress continuity conditions at interfaces. By relaxing these continuity conditions, the debonding effect can be accommodated in the theory.

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APPENDIX A

For future use in subsequent studies we write the stress constitutive equations in matrix form

$$\begin{Bmatrix} \sigma^1 \\ \sigma^2 \end{Bmatrix} = \begin{bmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{bmatrix} \begin{Bmatrix} e^1 \\ e^2 \end{Bmatrix} - \begin{bmatrix} p^{11} & p^{21} \\ p^{12} & p^{22} \end{bmatrix} \begin{Bmatrix} \theta^1 \\ \theta^2 \end{Bmatrix}, \quad (A1)$$

where σ^a and e^a are vector representations of stress and strain tensors defined by

$$\begin{aligned}\sigma^a &= (\sigma_{11}^a, \sigma_{22}^a, \sigma_{33}^a, \sigma_{12}^a, \sigma_{13}^a, \sigma_{23}^a) \\ e^a &= (e_{11}^a, e_{22}^a, e_{33}^a, 2e_{12}^a, 2e_{13}^a, 2e_{23}^a).\end{aligned}$$

C^{ab} is a 6×6 material coefficient matrix of the form

$$C^{ab} = \begin{bmatrix} C_{11}^{ab} & \dots & C_{16}^{ab} \\ \vdots & & \vdots \\ C_{61}^{ab} & & C_{66}^{ab} \end{bmatrix}.$$

and p^{ab} is a six dimensional vector defined by

$$p^{ab} = (p_{11}^{ab}, p_{22}^{ab}, p_{33}^{ab}, p_{12}^{ab}, p_{13}^{ab}, p_{23}^{ab}).$$

The stress constitutive equations (A1), govern the anisotropic behavior of a given composite. The symmetry conditions (eqn 3.17), indicate that the matrices C^{11} and C^{22} are both symmetric and C^{21} is equal to the transpose of C^{12} i.e. $C^{21} = C^{12T}$.

This establishes the symmetry of the overall stiffness matrix in eqn (A1).

The matrix form of the positive definiteness condition (4.3), is

$$\tilde{S}^T C S \geq 0, \quad (A2)$$

where S is an arbitrary twelve dimensional vector and C is the 12×12 overall stiffness matrix defined by

$$C = \begin{bmatrix} C^{11} & C^{12} \\ C^{12T} & C^{22} \end{bmatrix}, \quad (A3)$$

From linear algebra we know that the inequality (A2) is satisfied if all of the eigenvalues or principal minors of the symmetric matrix C are positive. This explicitly determines the constraints to be satisfied by the model constants.